Blow-up of Solutions for the Damped Boussinesq Equation

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We consider the blow-up of solutions as a function of time to the initial boundary value problem for the damped Boussinesq equation. Under some assumptions we prove that the solutions with vanishing initial energy blow up in finite time.

Key words: Damped Boussinesq Equation; Blow-up of Solutions; Initial Boundary Value Problems; Sobolev-Poincaré Inequality; Hölder Inequality.

1. Introduction

In this paper, we study the blow-up of solutions in the course of time for the following initial boundary value problem of the damped Boussinesq equation

$$u_{tt} - bu_{xx} + \delta u_{xxxx} - ru_{xxt} = (f(u))_{xx}, \quad (x,t) \in (0,1) \times (0,+\infty), \tag{1}$$

$$u(0,t) = u(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = 0, \quad t > 0, \tag{2}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in (0,1), \tag{3}$$

where b, δ, $r \geq 0$ are constants and f is a given nonlinear function with $f(0) = 0$.

Scott Russell's study [1] of solitary water waves motivated to analyse the development of nonlinear partial differential equations for modeling wave phenomena in fluids, plasmas, elastic bodies, etc. Especially, when $r = 0$, $b = 1$ and $f(u) = u^2$, (1) becomes the Boussinesq equation

$$u_{tt} - u_{xx} + \delta u_{xxxx} = (u^2)_{xx},$$

which is an important model which approximately describes the propagation of long waves on shallow water like the other Boussinesq equations (with u_{xxtt}, instead of u_{xxxx}). This equation was first deduced by Boussinesq [2]. In the case $\delta > 0$ this equation is linearly stable and allows small nonlinear transverse oscillations of an elastic beam (see [3] and references therein). It is called the "good" Boussinesq equation, while the equation with $\delta < 0$ received the name of the "bad" Boussinesq equation since it possesses a linear instability.

There is considerable mathematical interest in the Boussinesq equations which have been studied from various aspects (see [4 – 6] and references therein). Much efforts have been made to establish sufficient conditions for the nonexistence of global solutions to various associated boundary value problems [5, 7].

Levine and Sleeman [7] studied the global nonexistence of solutions for the equation

$$u_{tt} - u_{xx} - 3u_{xxxx} + 12(u^2)_{xx} = 0,$$

with periodic boundary conditions. Turitsyn [5] proved the blow-up in the Boussinesq equations

$$u_{tt} - u_{xx} + u_{xxxx} + (u^2)_{xx} = 0$$

and

$$u_{tt} - u_{xx} - u_{xxtt} + (u^2)_{xx} = 0$$

for the case of periodic boundary conditions and obtained exact sufficient criteria of the collapse dynamics. The generalization of Boussinesq equation

$$u_{tt} - u_{xx} + \alpha u_{xxxx} + f(u)_{xx} = 0$$

was studied in numerous papers [8 – 15].

Liu [12, 13] studied the instability of solitary waves for a generalized Boussinesq type equation

$$u_{tt} - u_{xx} + (f(u) + u_{xx})_{xx} = 0,$$

and established some blow-up results for a nonlinear Pochhammer-Chree equation

$$u_{tt} - u_{xxtt} - f(u)_{xx} = 0.$$

Zhijian [14, 15] studied the blow-up of solutions to the initial boundary value problems for the Boussinesq equations

$$u_{tt} - u_{xx} - bu_{xxxx} = \sigma(u)_{xx}$$

and

$$u_{tt} - u_{xx} - u_{xxtt} = \sigma(u)_{xx}.$$

Godefroy [8] showed the blow-up of the solutions for the following problem

$$u_{tt} = f(u)_{xx} + u_{xxtt}, \quad x \in R, \quad t \geq 0,$$
$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x),$$
$$u(+\infty,t) = u(-\infty,t) = 0,$$

where $f\colon R \to R, C^\infty,\ f(0) = 0$.

In the Boussinesq equations, the effect of small nonlinearity and dispersion is considered, but in many real situations, the damping effect is strong compared with the nonlinear and dispersive one. Therefore the damped Boussinesq equation

$$u_{tt} - 2bu_{txx} = -\alpha u_{xxxx} + u_{xx} + \beta(u^2)_{xx}$$

is considered as well, where u_{txx} is the damping term and α, $b = \text{const} > 0$, $\beta = \text{const} \in R$, (see [3, 4, 6, 16] and references therein).

Varlamov [3, 6] investigated the long-time behavior of solutions with given initial value, spatially periodic, and also initial-boundary value problems for the damped Boussinesq equation in two space dimensions.

In this paper, we establish a blow-up result for solutions with vanishing initial energy of the initial boundary value problem for the damped Boussinesq equation (1) – (3).

2. Blow-up of Solution

We first prove the following lemma.

Lemma. If there exist functions $\upsilon_0(x)$, $w_0(x) \in H^2$ such that the initial values $u_0(x)$ and $u_t(x,0)$ satisfy the relations

$$u(x,0) = (w_0(x))_x, \quad u_t(x,0) = (\upsilon_0(x))_x$$

then for all $t \in T$, the solution $u(x,t)$ of (1) – (3) satisfies $u(x,t) = (w(x,t))_x$, with a corresponding evolution of $w(x,t)$, $\upsilon(x,t)$ satisfying the system

$$w_t = \upsilon, \quad \upsilon_t = \upsilon_{xx} + w_{xx} - w_{xxxx} + (f(w_x))_x. \quad (4)$$

Proof. Writing (1) in the form

$$u_t = z_x, \quad z_t - rz_{xx} = (bu - \delta u_{xx} + f(u))_x \quad (5)$$

and integrating the first of these equations, we obtain $u(x,t) = u(x,0) + \int_0^t z_x(x,s)\mathrm{d}s$. The term $u(x,0)$ is an x-derivative by hypothesis and $\int_0^t z_x(x,s)\mathrm{d}s$ is an x-derivative. Therefore there exists a $w(x,t)$ such that $u(x,t) = (w(x,t))_x$, which gives, from (1),

$$u = w_x,\ w_{tt} - bw_{xx} + \delta w_{xxxx} - rw_{xxt} = (f(w_x))_x. \quad (6)$$

Hence we easily obtain the system (4).

By the Lemma the initial boundary value problem (1) – (3) corresponds the following problem

$$\begin{aligned} &w_{tt} - bw_{xx} + \delta w_{xxxx} - rw_{xxt} = (f(w_x))_x, \\ &(x,t) \in (0,1) \times (0,+\infty), \end{aligned} \quad (7)$$

$$\begin{aligned} &w_x(0,t) = w_x(1,t) = w_{xxx}(0,t) = w_{xxx}(1,t) = 0, \\ &t > 0, \end{aligned} \quad (8)$$

$$w(x,0) = w_0(x),\ w_t(x,0) = w_1(x),\ x \in (0,1). \quad (9)$$

Theorem. Let u be the solution of the problem (1) – (3). Assume that the following conditions are valid:

(i) $f(s) \in C(R)$, $f(s)s \leq k\int_0^s f(\tau)\mathrm{d}\tau \leq -k\beta|s|^{m+1}$, $s \in R$, where $k > 2$ and $\beta > 0$ are constants, also $1 < m \leq 3$.

(ii) $u(x,0) = (w_0(x))_x$, $u_t(x,0) = (\upsilon_0(x))_x$ for some $w_0 \in H_0^2(0,1)$ and $\upsilon_0 \in L_2(0,1)$ such that

$$E(0) = \frac{1}{2}\|\upsilon(0)\|_2^2 + \frac{b}{2}\|w_x(0)\|_2^2 + \frac{\delta}{2}\|w_{xx}(0)\|_2^2 + \int_0^1\int_0^{w_{0x}} f(s)\mathrm{d}s\mathrm{d}x \leq 0.$$

Then the solution u blows up in finite time

$$T \leq \begin{cases} \left[t_1^{\frac{3-m}{2}} + \dfrac{3-m}{2C_5(\alpha-1)y^{\alpha-1}(t_1)}\right]^{\frac{2}{3-m}}, & m < 3, \\ t_1 \exp\dfrac{1}{C_5(\alpha-1)y^{\alpha-1}(t_1)}, & m = 3, \end{cases}$$

where t_1 and y will be defined respectively by (22) and (23), C_8 and $\alpha > 1$ are constants to be defined later.

Proof. By multiplying (7) by w_t and integrating the new equation in the interval (0,1) then we obtain

$$\begin{aligned} &\int_0^1 w_t w_{tt}\mathrm{d}x - b\int_0^1 w_t w_{xx}\mathrm{d}x + \delta\int_0^1 w_t w_{xxxx}\mathrm{d}x \\ &-r\int_0^1 w_t w_{xxt}\mathrm{d}x = \int_0^1 w_t(f(w_x))_x\mathrm{d}x, \\ &E'(t) + r\|w_{xt}(t)\|_2^2 = 0,\ E(t) \leq E(0) \leq 0,\ t \geq 0, \end{aligned} \quad (10)$$

where

$$E(t)=\frac{1}{2}\|w_t(t)\|_2^2+\frac{b}{2}\|w_x(t)\|_2^2+\frac{\delta}{2}\|w_{xx}(t)\|_2^2 + \int_0^1\int_0^{w_x} f(s)\mathrm{d}s\mathrm{d}x, \quad t\geq 0. \tag{11}$$

Let

$$F(t)=\|w(t)\|_2^2+\int_0^t \|w_x(\tau)\|_2^2\mathrm{d}\tau, \tag{12}$$

then

$$F'(t)=2(w,w_t)+r\|w_x(t)\|_2^2, \tag{13}$$

$$\begin{aligned} F''(t) &= 2\Big(\|w_t(t)\|_2^2-b\|w_x(t)\|_2^2-\delta\|w_{xx}(t)\|_2^2-\int_0^1 w_x f(w_x)\mathrm{d}x\Big)\\ &\geq 2\Big(\|w_t(t)\|_2^2-b\|w_x(t)\|_2^2-\delta\|w_{xx}(t)\|_2^2-k\int_0^1\int_0^{w_x} f(s)\mathrm{d}s\mathrm{d}x\Big)\\ &\geq 2\Big(2\|w_t(t)\|_2^2-(k-2)\int_0^1\int_0^{w_x} f(s)\mathrm{d}s\mathrm{d}x-2E(0)\Big)\\ &\geq 2\left(2\|w_t(t)\|_2^2+(k-2)\beta\|w_x(t)\|_{m+1}^{m+1}-2E(0)\right), \quad t>0, \end{aligned} \tag{14}$$

where the assumption (i) of the Theorem and the fact that

$$k\int_0^1\int_0^{w_x} f(s)\mathrm{d}s\mathrm{d}x \leq 2E(0)-\|w_t(t)\|_2^2-b\|w_x(t)\|_2^2-\delta\|w_{xx}(t)\|_2^2+(k-2)\int_0^1\int_0^{w_x} f(s)\mathrm{d}s\mathrm{d}x$$

have been used. Taking the inequality (14) and integrating this, we obtain

$$F'(t)\geq 2(k-2)\beta\int_0^t \|w_x(\tau)\|_{m+1}^{m+1}\mathrm{d}\tau-4E(0)t+F'(0), t>0. \tag{15}$$

After this calculation, we could add the inequalities (14) with (15). We then get

$$\begin{aligned} F''(t)+F'(t) &\geq 4\|w_t(t)\|_2^2+2(k-2)\beta\left(\|w_x(t)\|_{m+1}^{m+1}+\int_0^t\|w_x(\tau)\|_{m+1}^{m+1}\mathrm{d}\tau\right)-4E(0)(1+t)+F'(0)\\ &= g(t), \quad t>0. \end{aligned} \tag{16}$$

Take $p=\frac{m+3}{2}$, obviously $2<p<m+1$ and $p'=\frac{m+3}{m+1}(<2)$. By using the Young inequality and the Sobolev-Poincaré inequality, we get

$$\begin{aligned} |(w,w_t)| &\leq \frac{1}{p}\|w(t)\|_p^p+\frac{1}{p'}\|w(t)\|_{p'}^{p'}\\ &\leq C_1\left[\left(\|w_x(t)\|_{m+1}^{m+1}\right)^{\mu}+\left(\|w_t(t)\|_2^2\right)^{\mu}\right], \end{aligned} \tag{17}$$

$$|(w,w_t)|^{\frac{1}{\mu}}\leq C_2\left[\|w_x(t)\|_{m+1}^{m+1}+\|w_t(t)\|_2^2\right], \; t>0,$$

where in this inequality and in the sequel C_i ($i=1,2,\ldots$) denote positive constants independent of t, $\mu=\frac{m+3}{2(m+1)}(<1)$. By the Sobolev-Poincaré inequality and the Hölder inequality

$$\|w_x(t)\|_{m+1}^{m+1}\geq\left(\|w(t)\|_2^2\right)^{\frac{m+1}{2}}, \quad t>0, \tag{18}$$

$$\|w_x(t)\|_{m+1}^{m+1}\geq\left(\|w_x(t)\|_2^2\right)^{\frac{m+1}{2}}, \quad t>0, \tag{19}$$

$$\int_0^t\|w_x(\tau)\|_{m+1}^{m+1}\mathrm{d}\tau\geq t^{\frac{1-m}{2}}\left(\int_0^t\|w_x(\tau)\|_2^2\mathrm{d}\tau\right)^{\frac{m+1}{2}}, \quad t>0, \tag{20}$$

By using the inequalities (17)–(20), we obtain

$$\begin{aligned} g(t) &\geq C_3\left(3\|w_x(t)\|_{m+1}^{m+1}+\|w_t(t)\|_2^2+\int_0^t\|w_x(\tau)\|_{m+1}^{m+1}\mathrm{d}\tau\right)-4E(0)t+F'(0)\\ &\geq C_4\left(|(w,w_t)|^{\frac{1}{\mu}}+\left(\|w(t)\|_2^2\right)^{\frac{m+1}{2}}+\left(\|w_x(t)\|_2^2\right)^{\frac{m+1}{2}}+t^{\frac{1-m}{2}}\left(\int_0^t\|w_x(\tau)\|_2^2\mathrm{d}\tau\right)^{\frac{m+1}{2}}\right)-4E(0)t+F'(0)\\ &\geq C_5 t^{\frac{1-m}{2}}\left(|(w,w_t)|^{\alpha}+\left(\|w(t)\|_2^2\right)^{\alpha}+\left(\|w_x(t)\|_2^2\right)^{\alpha}+\left(\int_0^t\|w_x(\tau)\|_2^2\mathrm{d}\tau\right)^{\alpha}\right)-4E(0)t+F'(0)-C_5t^{\frac{1-m}{2}},\\ &\quad t\geq 1, \end{aligned} \tag{21}$$

where in this inequality and in the following $\alpha = \frac{1}{\mu} > 1$. Since $-4E(0)t + F'(0) - C_5 t^{\frac{1-m}{2}} \to \infty$ as $t \to \infty$, there must be a $t_1 \geq 1$ such that

$$-4E(0)t + F'(0) - C_5 t^{\frac{1-m}{2}} \geq 0 \text{ as } t \geq t_1. \quad (22)$$

Let

$$y(t) = F'(t) + F(t), \quad (23)$$

then from the inequality (15) and the equality (12) we obtain $y(t) > 0$ as $t \geq t_1$. By using the inequality,

$$(a_1 + \ldots + a_l)^n \leq 2^{(n-1)(l-1)}(a_1^n + \ldots + a_l^n),$$

where $a_i \geq 0$ $(i = 1, \ldots, l)$ and $n > 1$ are reel numbers. Using the inequalities (22) and (21), we get

$$g(t) \geq C_5 t^{\frac{1-m}{2}} y^{\alpha}(t), \quad t \geq t_1. \quad (24)$$

So combining (16) with (24) gives

$$y'(t) \geq C_5 t^{\frac{1-m}{2}} y^{\alpha}(t), \quad t \geq t_1. \quad (25)$$

Therefore there exists a positive constant

$$T \leq \left[t_1^{\frac{3-m}{2}} + \frac{3-m}{2C_5(\alpha-1)y^{\alpha-1}(t_1)} \right]^{\frac{2}{3-m}}, \text{ for } m < 3,$$

and

$$T \leq t_1 \exp \frac{1}{C_5(\alpha-1)y^{\alpha-1}(t_1)}, \text{ for } m = 3, \quad (26)$$

such that

$$y(t) \to \infty \text{ as } t \to T^-. \quad (27)$$

By using (12), (13) and (27), we find

$$\begin{aligned} &2\|w(t)\|_2^2 + \|w_t(t)\|_2^2 + \|w_x(t)\|_2^2 \\ &+ \int_0^t \|w_x(\tau)\|_2^2 \mathrm{d}\tau \\ &\geq F'(t) + F(t) \to \infty \text{ as } t \to T^-. \end{aligned} \quad (28)$$

So (28) implies

$$\|w(t)\|_2^2 + \|w_t(t)\|_2^2 + \int_0^t \|w_x(\tau)\|_2^2 \mathrm{d}\tau \to \infty \text{ as } t \to T^-.$$

This completes the proof.

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